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PROPERTIES OF THE EXACT WAVE-TYPE SOLUTIONS IN THE THEORY OF STRATIFIED FLOWS†

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The general properties of the wave-type solutions in the theory of internal waves for flows in continuously stratified media are analysed. In addition to the well-known cases of the equivalence of the conditions for the summation of plane non-linear periodic waves and the principle of the superposition of linear waves, the conditions for the existence of wave-type solutions for non-stationary and attached waves in dissipative media are determined. The sets of relations of the physical parameters which can be used as expansion parameters when constructing approximate (asymptotic) solutions of the equations of internal waves in dissipative media are determined.

When constructing analytical solutions of problems of the motion of stratified media, simplified assumptions are widely used, such as the Boussinesq approximation and linearization of the equations themselves [1, 2] and of the boundary conditions [3]. The conditions of weak stratification and low viscosity enable the initial equations to be reduced to a linear singularly perturbed system. In some cases it is even possible to construct solutions of non-linear problems of the theory of wave generation, taking into account their interaction with the boundary flows [4].

In the approximation of an ideal fluid in an unbounded medium, solutions in the form of two-dimensional internal waves satisfy both the linear and non-linear equations [5]. In salinity – stream function variables it is possible to give an exact description of steady-state two-dimensional flows of an ideal fluid (in the general formulation for a linearly stratified medium and in the Boussinesq approximation for an exponentially stratified medium) in a channel of finite depth [6]. The existence of wave-type solutions considerably extends the possibilities of the analysis, and facilitates a comparison with laboratory experiments [7] and observations made in the ocean and atmosphere [8, 9].

It is of practical interest to determine the conditions for the existence of exact solutions for non-linear dissipative systems. In the latter case internal waves coexist with periodic boundary layers on solid surfaces and with internal boundary flows at discontinuities of the buoyancy frequency and its derivatives.‡ The purpose of the present paper is to analyse the conditions for wave-type solutions to exist, which convert the initial non-linear equations of motion of continuously stratified fluids into linear equations.

1. THE EQUATIONS OF MOTION

The system of equations of motion of a single-component incompressible stably stratified fluid with density $\rho(z)$ in a gravitational force field with an acceleration due to gravity $\mathbf{g} = -g\mathbf{e}_z$ in the Boussinesq approximation [10] has the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla P + \nu\Delta\mathbf{u} + \mathbf{g}S, \quad S_t + (\mathbf{u}\nabla)S = \varkappa_S \Delta S + w/\Lambda, \quad \operatorname{div}\mathbf{u} = 0 \tag{1.1}$$

Here $\mathbf{u} = \{u, \text{v}, w\}$ is the velocity, P is the dynamic component of the pressure, $\Lambda|\partial\ln\rho/\partial z|^{-1}$ is the stratification scale, t is the time, ν is the kinematic viscosity and $\varkappa_s$ is the diffusion coefficient of the salt. The total salinity S^* usually includes a time-independent stratification component and a dynamic component, and in a Cartesian system of coordinates $\{x, y, z\}$ the z axis is directed vertically upwards.

A fluid that is weakly stratified along the vertical is characterized by an equation of state of the form

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$\rho = \rho_0(1 - z/\Lambda + S(\mathbf{r}, t))$, where S is the dynamic component of the salinity and $\mathbf{r} = \{x, y, z\}$ is the radius vector. The initial and boundary conditions of system (1.1) include the condition for perturbations to decay at infinity, and also the no-slip condition for the velocity and the impermeability condition for the material on the bounding surfaces.

2. PLANE WAVES

In the two-dimensional case, using the stream function ψ, system (1.1) reduces to the following system of two equations

$$\begin{gathered}\Delta\psi_t + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y = \nu\Delta^2\psi + gS_x \\ S_t + \psi_y S_x - \psi_x S_y = \varkappa_S \Delta S - \psi_x / \Lambda\end{gathered} \tag{2.1}$$

In accordance with the formulation of the problem, the solution of system (2.1) will be sought in the form of the sum of wave-type invariant solutions

$$\psi = \sum \psi_i(\vartheta_i); \quad S = \sum S_i(\vartheta_i) \tag{2.2}$$

where $\vartheta_i = a_i x + b_i y - \omega_i t$ is the eikonal, and the components of the wave vector $\{a_i, b_i\}$ and the frequencies of the harmonics ω_i are assumed to be constant; here and henceforth summation with respect to i and/(or) j is carried out from 1 to N.

Substituting expression (2.2) into system (2.1) we obtain

$$\begin{gathered}-\sum \omega_i c_i^2 \psi_i''' + \sum \sigma_{ij} c_j^2 \psi_i' \psi_j''' = g \sum a_i S_i' \\ -\sum \omega_i S_i' + \sum \sigma_{ij} \psi_i' S_j' = -\gamma \sum a_i \psi_i'; \quad c_i^2 = a_i^2 + b_i^2\end{gathered} \tag{2.3}$$

where $\sigma_{ij} = b_i a_j - a_i b_j$ are the components of the second-rank antisymmetric tensor $\sigma_{ij} = -\sigma_{ji}$, $\sigma_{ii} = 0$, and the primes denote a derivative with respect to the corresponding variable with subscript i or j, denoting the order number of the function and of the independent variable. By taking these properties of the tensor $\|\sigma_{ij}\|$ into account we can rewrite system (2.3) in the equivalent form

$$\begin{gathered}-\sum \omega_i c_i^2 \psi_i''' + \sum_{i>j} \sigma_{ij}(c_j^2 \psi_i' \psi_j''' - c_i^2 \psi_j' \psi_i''') = g \sum a_i S_i' \\ -\sum \omega_i S_i' + \sum_{i>j} \sigma_{ij}(\psi_i' S_j' - \psi_j' S_i') = -\gamma \sum a_i \psi_i'\end{gathered} \tag{2.4}$$

Since only derivatives of the required functions occur in (2.4), the properties of the linear and non-linear parts of the solution can be analysed independently. The parts of the solution are separated using the substitution.

$$\psi_i = \psi_i^0 \vartheta_i + \psi_i^*, \quad S_i = S_i^0 \vartheta_i + S_i^* \tag{2.5}$$

(the asterisks will henceforth be omitted in the formulae). When the following relations are satisfied

$$c_j^2 \psi_i' \psi_j''' = c_i^2 \psi_j' \psi_i''', \quad \psi_i' S_j' = \psi_j' S_i' \tag{2.6}$$

the non-linear components of system (2.4) vanish identically. The condition for the separation of the variables converts Eqs (2.6) into the following system of ordinary differential equations

$$c_i^2 \psi_i''' = \lambda \psi_i', \quad S_i' = \mu \psi_i' \tag{2.7}$$

the particular solutions of which have the form

$$\psi_i = S_i / \mu = \psi_i^1 \exp(\vartheta_i \sqrt{\lambda} / c_i) + \psi_i^2 \exp(-\vartheta_i \sqrt{\lambda} / c_i) \tag{2.8}$$

where

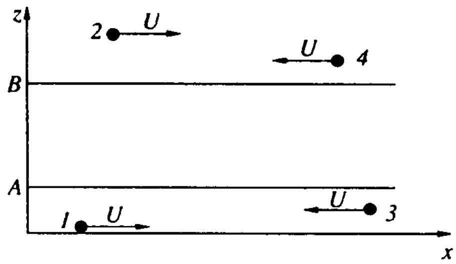

Fig. 1

$$\lambda(a_i x + b_i y)/c_i = \lambda(x\cos\alpha_i + y\sin\alpha_i)$$

Hence, the sum of the squares of the components of the wave vector c_i^2 turns out to be an invariant of the wave beam (2.8)

$$a_i^2 + b_i^2 = \lambda = k_0^2 \tag{2.9}$$

Direct substitution of expressions (2.8) into the initial system of equations (2.1) shows that the non-linear terms of the latter vanish identically.

It follows from the general properties of wave motions that, in addition to condition (2.9), an additional relation exists between the components of the wave vector and the frequencies of the components of the wave beam, imposed by dispersion relation [1], the specific form of which depends on the density distribution, the velocity of the flow and the dissipative properties of the fluid $\omega = \omega(k, U, \nu, \varkappa_s)$. For periodic waves and a linearly stratified medium at rest $\omega = \omega(k, \nu, \varkappa_s)$‡ and for associated waves in an ideal fluid $\omega = \omega_0 = \mathbf{k}\,\mathbf{U}$, where $\mathbf{U}$ = const is the velocity of motion of the source (the flow of fluid at infinity).

Below, we consider as an example the two-dimensional wave-type solutions of the linearized equations of a viscous stratified fluid, for which the following dispersion relations [3] holds

$$\omega^2(a^2 + b^2) - N^2 a^2 + i\nu\omega(a^2 + b^2)^2 = 0 \tag{2.10}$$

Condition (2.9), which ensures that the non-linear terms vanish, together with relation (2.10), form a system of two algebraic equations in three variables. Using this underdefined system, any two unknowns can be expressed in terms of the third with parameter k_0 ($b = b(a, k_0)$, $\omega = \omega(a, k_0)$). The relations thereby obtained, as will be shown below, are equivalent to the conditions for the superposition of the attached waves of arbitrary amplitude (including non-infinitesimal), which are produced by several independent sources moving with a velocity of the same modulus outside the empty zone (A, B) (see Fig. 1).

If the expression for the stream function in a space free from wave sources is written in the form of an expansion in the Fourier integral

$$\Psi = \int_{-\infty}^{+\infty} A(a)\exp[iax + ib(a)z - i\omega(a)t]da \tag{2.11}$$

the following relations hold for the solutions of system of equations (2.9), (2.10)

$$b(a) = \pm\sqrt{k_0^2 - a^2}, \quad \omega(a) = -i\nu k_0^2/2 \pm \sqrt{N^2 a^2/k_0^2 + \nu^2 k_0^4/4} \tag{2.12}$$

In a viscous fluid these solutions describe unsteady motions which decay with time. In an ideal fluid $\nu = 0$, and general expression (2.11) for the stream function, taking relations (2.12) into account, reduces to the form

‡See the previous footnote.

$$\Psi = I_r^+ + I_r^- + I_l^+ + I_l^- \tag{2.13}$$

$$I_q^\pm = \int_{-\infty}^{+\infty} A_q^\pm(a)\exp\left[iaX_q \mp \sqrt{k_0^2 - a^2}\, z\right]da, \quad q = r, l;$$

$$X_r = x - Ut, \quad X_l = x + Ut, \quad k_0 = \frac{N}{U}$$

where one or several of the coefficients A^+ and A^- may vanish independently.

The first (third) integral in (2.13) describes the field of a source moving to the right (to the left) and situated above the zone ($z < A$), while the second (fourth) integral describes the field of a source moving to the right (to the left) and situated above the zone ($z > b$). Hence, even in the non-linear formulation, the fields of several wave sources (in this case four, and in general any number) in an ideal fluid do not interact with one another and satisfy the superposition principle, established for infinitesimal waves.

This approach can be extended, by the methods of perturbation theory, to the case of a slightly viscous fluid, when the kinematic viscosity is chosen as the small parameter of the problem. In the general case, this approach also enables one to use other characteristics of the problem as the small parameter, including the amplitudes (slopes) of the waves and, as follows from relation (2.9), the degree of divergence of the squares of the wave vectors in the packet (i.e. the deviations of the sums of the squares of the wave numbers in the packet from k_0^2), and also their combinations.

Expressions (2.8) and (2.9) specify the form of the dispersion relation when calculating the pattern by which the waves build up and propagate, taking into account the start of the motion of the source in dissipative media. Examples of calculations of the build-up of infinitesimal internal waves for different laws of motion of singular sources were given in [11].

3. WAVES IN A MOVING MEDIUM

Problems with a finite value of the velocity of the fluid in a medium unperturbed by waves, including the theory of attached (lee) internal waves in a channel of finite depth, correspond to the general solutions of Eqs (2.7), containing linear functions of the eikonal with arbitrary coefficient $\psi_i = \psi_i^0\vartheta_i$, $S_i = S_i^0\vartheta_i$. Substitution of solutions of Eqs (2.7) converts system (2.4) into an overdetermined system of algebraic equations, where the equations

$$\lambda\sum\sigma_{ij}\psi_i^0\psi_j' = \sum(\lambda\omega_i + \mu g a_i)\psi_i', \quad \sum\sigma_{ij}(\mu\psi_i^0 - S_i^0)\psi_j' = \sum(\mu\omega_i - \gamma a_i)\psi_i' \tag{3.1}$$

correspond to the particular solutions (2.8), while the equations

$$\sum a_i S_i^0 = 0, \quad \sum\sigma_{ij}\psi_i^0 S_j^0 = \sum(\omega_i S_i^0 - \gamma a_i \psi_i^0). \tag{3.2}$$

correspond to the linear part of solution of Eqs (2.7).

For the further analysis it is convenient to rewrite relations (3.1) and (3.2) in vector-matrix notation

$$-\lambda A\psi^0\psi = (\lambda\omega + \mu g\mathbf{a})\boldsymbol{\psi}, \quad -A(\mu\psi^0 - \mathbf{S}^0)\boldsymbol{\psi} = (\mu\omega - \gamma\mathbf{a})\psi \tag{3.3}$$

$$\mathbf{a}\mathbf{S} = 0, \quad (A\mathbf{S}^0)\psi^0 = -(A\psi^0)\mathbf{S}^0 = \omega\mathbf{S}^0 - \gamma\mathbf{a}\psi^0 \tag{3.4}$$

Here A is an $N \times N$ antisymmetric matrix of the tensor $\|\sigma_{ij}\|$.

Subtracting the second equation of (3.3), multiplied by λ from the first equation of (3.3), multiplied by μ, and collecting terms of like components of the vector $\boldsymbol{\psi}$, we obtain an overdetermined system in the coefficients $\mathbf{S}^0$ and $\boldsymbol{\psi}^0$

$$-\lambda A\psi^0 = \lambda\omega + \mu g\mathbf{a}, \quad -A\mathbf{S}^0 = (\lambda\gamma + \mu^2 g)\mathbf{a}, \quad \mathbf{a}\mathbf{S}^0 = 0 \tag{3.5}$$

$$(A\mathbf{S}^0)\psi^0 = -(A\psi^0)\mathbf{S}^0 = \omega\mathbf{S}^0 - \gamma\mathbf{a}\omega^0$$

We conclude from a comparison of the first and third equations of (3.5), taking into account the last equation of the system, that

$$\mathbf{a}\psi^0 = 0 \quad \text{and} \quad \mathbf{S}^0 = \beta\psi^0 \tag{3.6}$$

i.e. the vectors $\mathbf{S}^0$ and ψ^0 are collinear. It follows from the second equation of (3.5) that the vectors ω and ψ^0 are also collinear.

Using the relations obtained and taking the first two equations of (3.5) into account, we find the relation between the frequencies $\boldsymbol{\omega}$ and the components of the wave vectors – the analogue of the dispersion relation corresponding to system (2.4)

$$-\lambda\beta A\psi^0 = \lambda\beta\omega + \mu\beta g\mathbf{a} = (\lambda\gamma + \mu^2 g)\mathbf{a} \Rightarrow \beta\lambda\omega = [\lambda\gamma + \mu(\mu - \beta)g]\mathbf{a}. \tag{3.7}$$

The remaining two equations ($-\lambda\beta A\psi^0 = (\lambda\gamma + \mu^2 g)\mathbf{a}$ and the fourth equation of (3.5)) form an over determined system of $N + 1$ linear algebraic equations for the N variables $\{S_i^0\}$. To solve it we will introduce an additional fictitious variable $S_{N+1}^0 = \gamma + \mu^2/g$, after which, for the new set of variables, we obtain a homogeneous system of $N + 1$ linear algebraic equations. Non-trivial solutions of this system exist if the determinant of the matrix A_{N+1} is equal to zero.

Taking into account the fact that the determinants of antisymmetric matrices equal zero for an odd number of rows and are non-zero otherwise, since $\det A = \det A^T = (-1)^n \det A$, we obtain that for an odd number of waves ($N = 2n-1$) in solution (2.5) there is no linear part, modelling the average flow in the medium. For an event number of waves N the linear part of solution (2.5) is defined uniquely. The solution constructed describes wave packets with circular polarization, for which all the components have the same sums of the squares of the wave numbers.

4. COLLINEAR WAVES

A wave packet containing only waves with parallel wave vectors (but, generally speaking, with different frequencies) corresponds to the degenerate case $\sigma_{ij} = 0$, and to vanishing non-linear terms. The general expression describing such waves has the form

$$f = f(kR, t) = \int \bar{f}(k, \omega)\exp[i(kR - \omega t)]dk d\omega \tag{4.1}$$

$$R = \cos\alpha_x x + \cos\alpha_y y + \cos\alpha_z z$$

where k is the modulus of the wave number and $\cos\alpha_x$, $\cos\alpha_y$, $\cos\alpha_z$ are direction cosines of the specified direction. An example of such waves is each of the beams of a "St Andrew Cross" of periodic internal waves, excited in a continuously stratified fluid by a compact linear source [3].

5. THREE-DIMENSIONAL WAVES

In three-dimensional problems, when the motion of an incompressible fluid is characterized by a vector potential $\mathbf{A} = \{q_1, q_2, q_3\}$ ($\mathbf{u} = \text{rot}\,\mathbf{A}$), non-linear terms in the momentum and mass transfer equations of system (1.1) are expressed in terms of $\mathbf{A}$ by the rules of vector analysis.

Using the same scheme as above, the wave solutions will be sought in the form

$$\mathbf{A} = \sum_i \mathbf{A}_i(z_i), \quad S = \sum_i S_i(z_i); \quad z_i = \mathbf{k}_i\mathbf{r} - \omega_i t, \quad \mathbf{A}_i = \{q_{1i}, q_{2i}, q_{3i}\} \tag{5.1}$$

For the non-linear terms we obtain

$$(\mathbf{u}\nabla)S = \text{rot}\,\mathbf{A}\cdot\nabla S = \sum S_i' M_{ij}, \quad (\text{rot}\,\mathbf{A}, \nabla)\Delta\mathbf{A} = \sum k_i^2 \mathbf{A}'''_i M_{ij}$$

$$(\text{grad}\,\text{div}\,\mathbf{A}, \nabla)\,\text{rot}\,\mathbf{A} - (\Delta\mathbf{A}, \nabla)\,\text{rot}\,\mathbf{A} = \sum (C_{ij}D_{jj} - k_j^2 D_{ij})(\mathbf{k}_i \times \mathbf{A}_i'') \tag{5.2}$$

$$(\text{rot}\,\mathbf{A}, \nabla)\,\text{grad}\,\text{div}\,\mathbf{A} = \sum M_{ij} D_{ii} \mathbf{k}_i$$

where $C_{ij} = (\mathbf{k}_i, \mathbf{k}_j)$, $D_{ij} = (\mathbf{k}_i, \mathbf{A}_j'')$, $M_{ij} = \langle \mathbf{k}_i, \mathbf{k}_j, A_j' \rangle$, and the angle brackets denote a mixed product of three vectors.

An analysis of expressions (5.2) shows that the non-linear terms of the equations of motion vanish identically in two cases: when the wave vectors of each component of a wave from the packet are parallel or perpendicular to the vector potential. The first case corresponds to the condition $\text{rot}\mathbf{A} = 0$ and describes a fluid at rest. The second case splits into two: with wave vectors ($\mathbf{k}_i \parallel \mathbf{k}_j$), collinear with the packet, when the solution is described by functions of the form (4.1), and with collinear components of the vector potential ($\mathbf{A}_i \parallel \mathbf{A}_j$), when the problem in fact becomes two-dimensional and the solutions have the form (2.8).

6. CONCLUSION

The above analysis shows that the previously mentioned linearization of the equations of motion for plane internal waves within a continuously stratified fluid [5] is a special case of a more general situation, which includes the possibility of the summation of not only periodic but also of attached (stationary and non-stationary) two-dimensional waves of finite amplitude, produced by several independent sources.

In the three-dimensional case the superposition principle remains valid only for waves with parallel wave vectors, which is satisfied for each of four wave beams, produced by a vertically oscillating source with an arbitrary oscillation law.

The proposed method not only enables one to obtain the conditions for the superposition principle to be satisfied for internal waves of arbitrary amplitude in an ideal fluid, but can also serve as a basis for constructing asymptotic solutions, describing waves of finite amplitudes in a viscous fluid with diffusion. Here the set of expansion parameters is supplemented by new small parameters, connected with the degree of divergence of the wave vectors in the beam, which enables calculations to be carried out on waves of arbitrary amplitude. The specific form of the expansion parameter in the latter case is determined by the geometry of the problem and the dissipative properties of the medium.

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